Chaotic laser–matter interaction

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Hamiltonian Silnikov orbits are shown to be produced when a system of two-level atoms interacting via a self-consistent electric field in a single-model lossless cavity is perturbed by a small-amplitude probe laser (a weak, externally imposed, monochromatic electromagnetic field). This result is obtained by utilizing the two constants of motion which exist in the perturbed problem and by calculating a Melnikov function whose nondegenerate zeros imply transverse intersections of the stable and unstable manifolds of a hyperbolic fixed point set. The dynamics of the system on the Silnikov orbits contains a Smale horsehoe construction, leading to intermittent switching of the sign of the electric field which has extreme sensitivity to initial conditions.

1. Introduction

Chaos in laser–matter interactions has been a topic of considerable interest over the past few years. Several studies [1–4] address the issue of chaotic time-dependence as atoms and molecules interact with self-consistently generated electromagnetic fields. These works consider Maxwell–Bloch dynamics in a generalization of the Jaynes–Cummings model [5]. In this model, an ensemble of two-state atoms interacts resonantly with a (classical) electromagnetic field in a single-mode lossless cavity. Averaging over the fast phases in the problem and neglecting nonresonant terms leads to a completely integrable Hamiltonian system which describes the Maxwell–Bloch dynamics in the so-called rotating wave approximation (RWA).

Numerical studies by Belobrov et al. [1,2], and by Milonni et al. [3], indicate that the nonresonant terms neglected in the RWA lead to chaotic behavior (e.g., extreme sensitivity to initial conditions), which is claimed to occur above a threshold value of the coupling constant between the atoms and the field. In contrast, the numerical study by Alekseev and Berman [4] indicates that external forcing perturbations may cause chaos even *within* the RWA. Alekseev and Berman [4] introduce a probe laser (an external monochromatic field of constant amplitude), which is detuned from the resonant laser–matter interaction. Even at small amplitude, the external forcing perturbation of the probe laser is sufficient to break the integrability of the RWA. The numerical calculations of Alekseev and Berman [4] indicate that chaotic behavior is caused by the probe laser perturbation, even for small values of the coupling constant between the atoms and the field. In this case, the numerical evidence is broad-band power spectra of time series. Alekseev and Berman [4] also give a heuristic and approximate analytical discussion of this chaos in terms of a perturbed pendulum model of the laser–matter dynamics.

In this paper, we present the exact solution for the problem treated numerically and approximately in ref. [4]. In particular, we show that homoclinic orbits of Hamiltonian Silnikov type are produced by the perturbation, and thus provide the Smale horseshoe mechanism responsible for chaos in the model. This establishes analytically that the Maxwell–Bloch laser–matter interaction equations in the RWA can exhibit chaotic behavior under small external forcing.

Within the RWA the dynamical system of perturbed Maxwell–Bloch equations is

$$\dot{\mathcal{E}}=\mathcal{P}\,,\quad \dot{\mathcal{P}}=(\mathcal{E}+\varepsilon e^{i\omega t})\mathcal{S}\,,$$
$$\dot{\mathcal{S}}=-\tfrac{1}{2}[(\mathcal{E}+\varepsilon e^{i\omega t})\mathcal{P}^{*}+(\mathcal{E}^{*}+\varepsilon e^{-i\omega t})\mathcal{P}]\,.$$

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The variables in this set of equations are dimensionless (see eq. (11) in ref. [4] for their dimensional form). In these equations overdots denote time derivatives, $\mathscr{E}$ denotes the self-consistent electric field, $\mathscr{P}$ is the polarizability of the matter, and $\mathscr{D}$ is the difference of its occupation numbers, assuming the material response may be modeled by two levels – a ground state and an excited state. Here, $\mathscr{E}$ and $\mathscr{P}$ are complex scalar functions of time, $\mathscr{D}$ is real, ε is the (constant) amplitude of the external driving field, and ω is the detuning of the laser probe frequency from resonance with the two-level atoms. For nonzero ε and ω, these equations possess two integrals of motion:

$$H=\tfrac{1}{2}|\mathscr{P}|^2+\tfrac{1}{2}\mathscr{D}^2\,,$$

$$L=\tfrac{1}{2}\omega|\mathscr{E}|^2+\omega\mathscr{D}$$

$$+\frac{1}{2\mathrm{i}}\left[(\mathscr{E}+\varepsilon \mathrm{e}^{\mathrm{i}\omega t})\mathscr{P}^*-(\mathscr{E}^*+\varepsilon \mathrm{e}^{-\mathrm{i}\omega t})\mathscr{P}\right]\,.$$

These two integrals of motion result from unitarity (H) and energy conservation (L). The three summands in L involve the self-consistent electric field energy, $\tfrac{1}{2}|\mathscr{E}|^2$, the excitation energy of the atoms, $\mathscr{D}$, and the interaction energy of the polarizable medium with the *total* electric field, $\mathscr{E}+\varepsilon\mathrm{e}^{\mathrm{i}\omega t}$. For a derivation of the perturbed Maxwell–Bloch equations from the Maxwell–Schrödinger equations and further discussion of the conserved quantities H and L, see ref. [6].

2. The unperturbed problem

The unperturbed Maxwell–Bloch equations, with $\varepsilon=0$, are

$$\dot{\mathscr{E}}=\mathscr{P}\,,\quad \dot{\mathscr{P}}=\mathscr{E}\mathscr{D}\,,\quad \dot{\mathscr{D}}=-\tfrac{1}{2}(\mathscr{E}\mathscr{P}^*+\mathscr{E}^*\mathscr{P})\,.$$

These unperturbed equations posses three integrals of motion:

$$H=\tfrac{1}{2}|\mathscr{P}|^2+\tfrac{1}{2}\mathscr{D}^2\,,\quad J=\frac{1}{2\mathrm{i}}(\mathscr{E}\mathscr{P}^*-\mathscr{E}^*\mathscr{P})\,,$$

$$K=\tfrac{1}{2}|\mathscr{E}|^2+\mathscr{D}\,.$$

That is, in the absence of the external forcing, unitarity still yields conservation of H; the interaction energy J is now separately conserved; and K, the sum of the electric field energy and the atomic excitation energy, is also conserved. (Note: when $\omega=0$ and $\varepsilon\neq 0$ there are also three conservation laws.) We eliminate $\mathscr{D}$ in favor of the conserved quantity K. Hence, the unperturbed equations become

$$\dot{\mathscr{E}}=\mathscr{P}\,,\quad \dot{\mathscr{P}}=\mathscr{E}(K-\tfrac{1}{2}|\mathscr{E}|^2)\,,\quad \dot{K}=0\,.$$

On each level surface of K, these equations restrict to the ideal complex Duffing equation, which can be derived from the Hamiltonian H in *canonical* phase space variables $\mathscr{E}$ and $\mathscr{P}$,

$$H=\tfrac{1}{2}|\mathscr{P}|^2+\tfrac{1}{2}(K-\tfrac{1}{2}|\mathscr{E}|^2)^2\,,$$

with Poisson bracket $\{\mathscr{E}^*,\mathscr{P}\}=2$. Upon writing $\mathscr{P}=\mathscr{P}_1+\mathrm{i}\mathscr{P}_2$ and $\mathscr{E}=\mathscr{E}_1+\mathrm{i}\mathscr{E}_2$, this Poisson bracket relation implies the usual form, $\{\mathscr{E}_i,\mathscr{P}_j\}=\delta_{ij}$; so $\mathscr{E}_1$ and $\mathscr{E}_2$ may be taken as two real phase-space coordinates, and $\mathscr{P}_1$ and $\mathscr{P}_2$ as their canonically conjugate momenta, in the usual symplectic sense. The other integral, J, may now be written as

$$J=\frac{1}{2\mathrm{i}}(\mathscr{E}\mathscr{P}^*-\mathscr{E}^*\mathscr{P})=\mathscr{E}_1\mathscr{P}_2-\mathscr{E}_2\mathscr{P}_1\,.$$

Hence, the interaction energy J acts as the angular momentum for the system in these variables. That is, upon using the canonical Poisson brackets above for $\mathscr{E}$ and $\mathscr{P}$, the quantity J generates an identical phase rotation in both $\mathscr{E}$, $\mathscr{P}$, namely, $J:(\mathscr{E},\mathscr{P})\to(\mathrm{e}^{-\mathrm{i}\varphi}\mathscr{E},\mathrm{e}^{-\mathrm{i}\varphi}\mathscr{P})$. This, of course, suggests transforming to polar coordinates.

We write $\mathscr{E}=Q\mathrm{e}^{\mathrm{i}\theta}$, where Q and θ are the new coordinates, and $P=\dot{Q}$ and $J=Q^2\dot{\theta}$ are their canonically conjugate momenta. The map from (Q,P,θ,J) to $(\mathscr{E},\mathscr{P})$ is given by

$$\mathscr{E}=Q\mathrm{e}^{\mathrm{i}\theta}\,,\quad \mathscr{P}=(P+\mathrm{i}J/Q)\mathrm{e}^{\mathrm{i}\theta}\,.$$

The Hamiltonian in the polar coordinates is independent of θ,

$$H=\tfrac{1}{2}P^2+\frac{J^2}{2Q^2}+\tfrac{1}{2}(K-\tfrac{1}{2}Q^2)^2\,,$$

and Hamilton's equations for (Q,P,θ,J) are

$$\dot{Q}=P\,,\quad \dot{P}=Q(K-\tfrac{1}{2}Q^2)+\frac{J^2}{Q^3}\,,$$

$$\dot{\theta}=\frac{J}{Q^2}\,,\quad \dot{J}=0\,.$$

From these equations, we obtain four basic Q–P phase portraits of this system, shown in figs. 1a–1d. The phase portrait of most interest here appears in fig. 1a, which shows a pair of homoclinic orbits in the Q–P plane for the case $J=0$ and $K>0$. The homoclinic orbits are absent for the case $J=0$ and $K<0$, shown in fig. 1b, and are also absent for the cases with $J\neq 0$ shown in fig. 1c ($K>0$) and fig. 1d ($K<0$). Figs. 1c and 1d show the usual "centrifugal barrier" singularity, introduced at $Q=0$ in the phase portraits for $J\neq 0$ by the transformation to polar coordinates (Q, P, θ, J). There is no singularity at $\mathscr{E}=0$ in the original $\mathscr{E}$–$\mathscr{P}$ coordinates.

Having used the polar coordinates to identify the homoclinic orbits present in the unperturbed problem, we return to the original $\mathscr{E}$–$\mathscr{P}$ coordinates. The homoclinic orbits in the phase portrait for the case $J=0$ and $K>0$ represent a "pinched" torus of orbits in the original $\mathscr{E}$–$\mathscr{P}$ coordinates homoclinic to the hyperbolic fixed point at the origin (see fig. 2). A particular solution on this torus is expressible in the complex $\mathscr{E}$–$\mathscr{P}$ phase space as

$$\mathscr{E}=2\sqrt{K}\,\mathrm{sech}(\sqrt{K}\,t)\,\mathrm{e}^{\mathrm{i}\theta}\,,$$

$$\mathscr{P}=-2K\,\mathrm{sech}(\sqrt{K}\,t)\tanh(\sqrt{K}\,t)\,\mathrm{e}^{\mathrm{i}\theta}\,.$$

The general solution on the homoclinic torus is obtained upon replacing t in this formula by $t-t_0$. The hyperbolic point at $\mathscr{E}=0=\mathscr{P}$ corresponds to all the atoms being initially in the excited state, with no field energy. This point is unstable, and the solution on the torus describes the exchange of energy between the two-level atoms and the self-consistent field that results from a perturbation along the unstable direction. Since the electric field and the polarizibility have the same phase along the homoclinic orbit (and this phase is constant for $J=0$) the motion along the homoclinic orbit is just the homoclinic motion of the real Duffing oscillator.

3. The perturbed problem

The discussion of the phase-space geometry of the unperturbed Maxwell–Bloch problem in the previous section now provides the framework for studying the perturbed problem. The quantity ωJ, an integral of motion for the unperturbed problem, may be used to generate a time-dependent canonical transformation to a frame which phase-rotates at the detuning frequency. The new complex canonical variables are

$$x=\mathscr{E}\mathrm{e}^{-\mathrm{i}\omega t}\,,\quad y=\mathscr{P}\mathrm{e}^{-\mathrm{i}\omega t}\,,$$

with Poisson bracket $\{x^*, y\}=2$. This transformation leaves K unchanged (since $\{J, K\}$ vanishes) and brings the perturbed Maxwell–Bloch equations into *autonomous* form, namely

$$\dot{x}=-\mathrm{i}\omega x+y\,,$$

$$\dot{y}=-\mathrm{i}\omega y+(x+\varepsilon)(K-\tfrac{1}{2}|x|^2)\,,$$

$$\dot{K}=-\tfrac{1}{2}\varepsilon(y+y^*)\,.$$

These equations with ε set equal to zero may be derived using the Poisson bracket $\{x^*, y\}=2$ and the Hamiltonian $\tilde{H}=H-\omega J$, namely

$$\tilde{H}=\tfrac{1}{2}|y|^2+\tfrac{1}{2}(K-\tfrac{1}{2}|x|^2)^2+\tfrac{1}{2}\mathrm{i}\omega(x^*y-y^*x)\,.$$

When ε is *nonzero*, the two constants of motion in

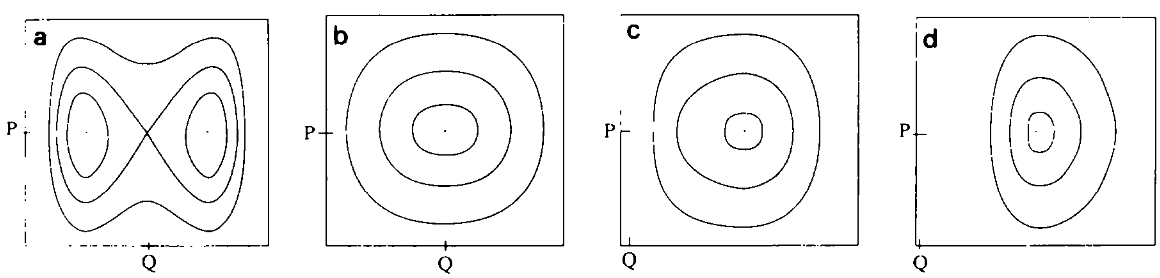

Fig. 1. (a) Q–P phase portrait of the unperturbed Maxwell–Bloch equations in polar coordinates for the case $J=0$ and $K>0$. (b) Q–P phase portrait for the case $J=0$ and $K<0$. (c) Q–P phase portrait for the case $J\neq 0$ and $K>0$. (d) Q–P phase portrait for the case $J\neq 0$ and $K<0$.

the first section are expressible in the phase-rotating frame as

$$H=\tfrac{1}{2}|y|^2+\tfrac{1}{2}(K-\tfrac{1}{2}|x|^2)^2\,,$$

from unitarity, and

$$L=\omega K+\frac{1}{2\mathrm{i}}\left[(x+\varepsilon)y^*-(x^*+\varepsilon)y\right],$$

the total energy.

Remarks

(1) For *any* value of ε, the conserved quantity L generates the autonomous perturbed Maxwell–Bloch equations as a Hamiltonian system, $\dot{G}=\{G,L\}'$, for $G\in\{x, x^*, y, y^*, K\}$ when using the *noncanonical* Poisson bracket, $\{\ ,\ \}'$, defined by the following relations and their complex conjugates,

$$\{x,x^*\}'=-2\mathrm{i}\,,\quad \{x,K\}'=-\mathrm{i}x\,,$$

$$\{y,y^*\}'=2\mathrm{i}(K-\tfrac{1}{2}|x|^2)\,,\quad \{y,K\}'=-\mathrm{i}y\,.$$

These noncanonical Poisson brackets satisfy the Jacobi identity, by virtue of being an invertible variable transformation of the symplectic bracket, $\{x, x^*\}=-2\mathrm{i}$, in direct sum with the Lie–Poisson bracket defined on su(2)*, the dual of the Lie algebra su(2). (Note: a Lie–Poisson bracket $\{G,H\}_{\mathrm{LP}}$ between real-valued functions G and H on the dual $\mathcal{G}^*$ of a Lie algebra $\mathcal{G}$ is defined as the linear functional on $\mathcal{G}$,

$$\{G,H\}_{\mathrm{LP}}=\langle\mu,[\partial G/\partial\mu,\partial H/\partial\mu]\rangle\,,$$

where $\mu\in\mathcal{G}^*$, $\partial G/\partial\mu$ and $\partial H/\partial\mu\in\mathcal{G}$, the bracket $[\ ,\ ]$ denotes the Lie product on Γ, and $\langle\ ,\ \rangle$ is a real-valued non-degenerate pairing between $\mathcal{G}^*$ and $\mathcal{G}$. For an introductory discussion of Lie–Poisson brackets, see ref. [8]. The phase-rotating Bloch variables y, y^*, and $\mathcal{D}$, satisfy the su(2)* Lie–Poisson bracket relations, $\{y,y^*\}_{\mathrm{LP}}=2\mathrm{i}\mathcal{D}$ and $\{y^*,\mathcal{D}\}_{\mathrm{LP}}=\mathrm{i}y^*$.) The quantity H is the Casimir function for the noncanonical Poisson bracket; that is, $\{G,H\}'=0$, for every function G on the set of variables $\{x, x^*, y, y^*, K\}$. So, H is preserved as a consequence of the degeneracy of the noncanonical Poisson bracket, while L is preserved by virtue of being the Hamiltonian for this bracket.

(2) Holm and Fordy [7] show that the unperturbed Maxwell–Bloch equations are tri-Hamiltonian: the unperturbed equations are expressible in Hamiltonian form using any of *three* inequivalent Lie–Poisson brackets. These three Lie–Poisson brackets are related to the Lax-pair formulation of the equations and the inverse spectral method for solving them.

We now investigate the equilibria of the perturbed Maxwell–Bloch equations in the phase-rotating frame. (Note: equilibrium solutions in the phase-rotating frame are *relative* equilibria; in the fixed frame these solutions are periodic orbits with period $2\pi/\omega$.) These equilibria occur at the zeros of the right hand sides of the equations for x, y, and K. For $\varepsilon=0$, the state $x=y=0$ is a hyperbolic equilibrium for all positive K. It is connected to itself by the two-dimensional "pinched" torus of homoclinic orbits found earlier in section 2 and given in the phase-rotating frame by

$$x=2\sqrt{K}\,\mathrm{sech}(\sqrt{K}\,\exp[\mathrm{i}(\theta_0-\omega t)]\,,$$

$$y=-2K\,\mathrm{sech}(\sqrt{K}\,t)\,\tanh(\sqrt{K}\,t)\,\exp[\mathrm{i}(\theta_0-\omega t)]\,.$$

In other words, for $\varepsilon=0$ the unperturbed Maxwell–Bloch equations possess a curve of equilibria at (0, 0, K) connected to itself in the x–y–K phase space by a homoclinic manifold. For each value of K, the homoclinic manifold restricts to the pinched torus formulas above (see fig. 3).

For nonzero ε, we look for a nearby curve of equilibria $x=x(K)$, $y=y(K)$ with $x(K)$, $y(K)$ being $\mathrm{o}(\varepsilon)$. We find

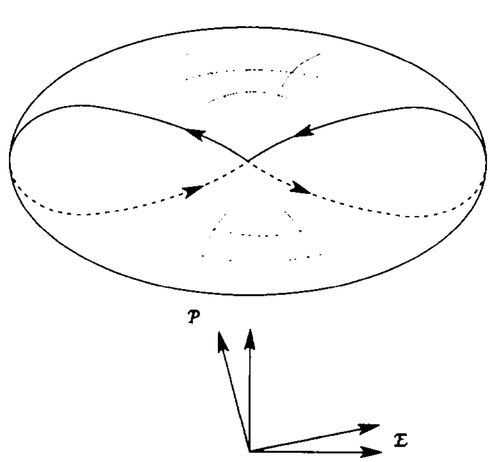

Fig. 2. Sketch of a pinched torus of homoclinic orbits. The central point, $\mathcal{E}=0=\mathcal{P}$, corresponds physically to all of the atoms being in the excited state.

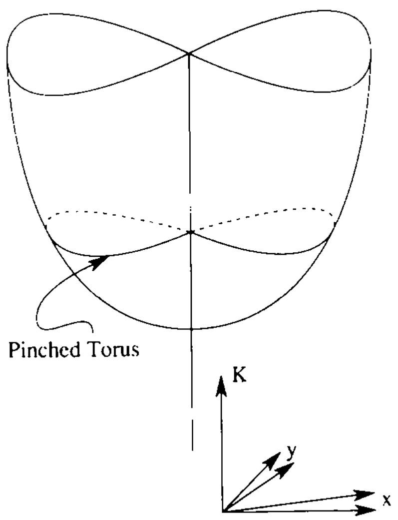

Fig. 3. Sketch showing the curve of hyperbolic equilibria and its homoclinic manifold, the union of the pinched tori parameterized by K.

$$y_1 = x_2 = 0\,, \quad y_2 = \omega x_1\,,$$

$$\omega^2 x_1 + (x_1 + \varepsilon)(K - \tfrac{1}{2}x_1^2) = 0\,.$$

By the implicit function theorem, this equation has a root $x_1 = \mathrm{o}(\varepsilon)$, and hence the required curve of equilibria persists. The eigenvalues at each of the equilibria are equal to

$$0,\ \pm\{[\sqrt{K} + \mathrm{o}(\varepsilon^2)] \pm \mathrm{i}[\omega + \mathrm{o}(\varepsilon^2)]\}\,.$$

The zero eigenvalue corresponds to the fact that the equilibria form a curve and so have a neutral direction (along K) due to the existence of the two constants of motion in the system, H and L. The form of the other four eigenvalues is due to the Hamiltonian nature of the problem. The orbits on both the stable and unstable manifolds passing through each equilibrium on the curve of equilibria are spirals.

4. Transverse intersections obtained by Melnikov zeros

For each choice of H and L, each equilibrium of the perturbed problem possesses a two-dimensional stable manifold, and a two-dimensional unstable manifold. To check for intersections of these stable and unstable manifolds, we make use of the unperturbed integrable structure. In particular, we use the "pinched" homoclinic tori of the unperturbed problem to parametrize the perturbed manifolds, and we measure distance between the perturbed manifolds of each perturbed equilibrium along the vectors normal to those tori. For a comprehensive treatment and references for this method, see ref. [9]. For more details of the present application, see ref. [6]. The unperturbed tori are either parametrized by the solutions stated above for them, or are given in implicit form by the equations

$$K = \text{const}\,, \quad H = \tfrac{1}{2}K^2\,, \quad J = 0\,.$$

Each homoclinic torus therefore possesses three normal vectors:

$$\boldsymbol{n}_1 = \nabla K\,, \quad \boldsymbol{n}_2 = \nabla(H - \tfrac{1}{2}K^2)\,, \quad \boldsymbol{n}_3 = \nabla J\,,$$

with

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial K}\right).$$

At first glance, it seems that we must measure the splitting distance in five dimensions along all three normals. However, the two-dimensional stable and unstable manifolds of any of the equilibria are *both* contained in the three-dimensional hypersurface $H = \text{const}$, $L = \text{const}$ (see fig. 4). Therefore, a distance measurement in only one direction is needed. We choose to measure it along $\boldsymbol{n}_3 = \nabla J$, using the Melnikov technique [9]. The Melnikov function measuring the distance between the stable and unstable manifolds of the curve of equilibria depending on K for $\varepsilon \neq 0$ is the following integral along the unperturbed homoclinic orbit,

$$M(\theta_0) = \int_{-\infty}^{\infty} \boldsymbol{n}_3 \cdot \boldsymbol{g}\, \mathrm{d}t$$

$$= \varepsilon\pi\omega^2 \operatorname{sech} \frac{\omega\pi}{2\sqrt{K}} \sin\theta_0\,,$$

where $\boldsymbol{g}$ is the perturbation part of the vector field,

$$\boldsymbol{g} = (0, 0, \varepsilon(K - \tfrac{1}{2}|x|^2), 0, -\varepsilon y_1)\,.$$

(Notice that $\boldsymbol{n}_3 \cdot \boldsymbol{g} = -\varepsilon\ddot{Q}\sin(\theta_0 - \omega t)$; so, after integrating by parts twice, the integral along the unperturbed orbit $Q(t)$ can be found in standard tables.)

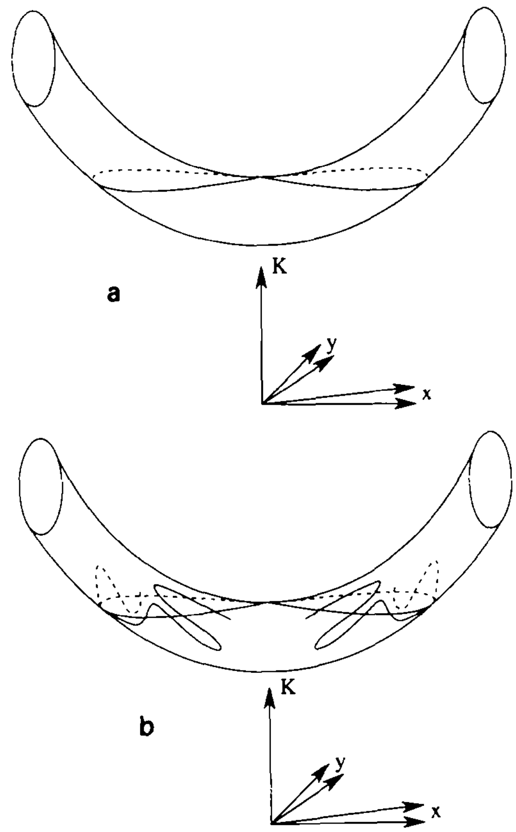

Fig. 4. (a) Sketch of an unperturbed hypersurface, H=const, L=const, showing a pinched homoclinic torus. (b) Sketch of transverse intersections on a perturbed hypersurface, H=const, L=const. (All of the intersections taken together on one arm of the sketch comprise one Hamiltonian Silnikov orbit).

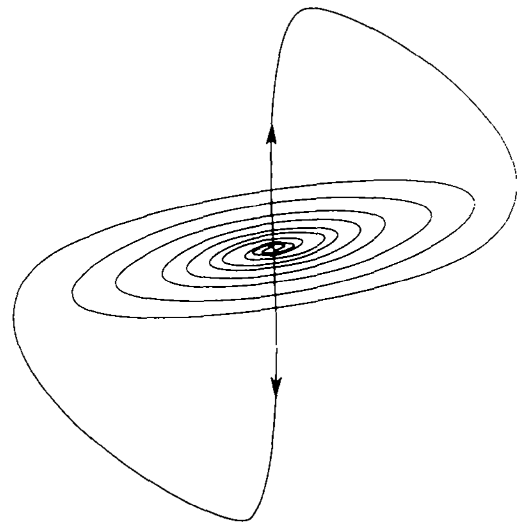

Fig. 5. Sketch of a pair of Hamiltonian Silnikov orbits. (The drawing actually shows a pair of regular Silnikov orbits. Hamiltonian Silnikov orbits would spiral inwards along the stable manifold and outwards along the unstable manifold in an intrinsically four-dimensional way; but this, of course, is impossible to draw!)

The Melnikov function $M(\theta_0)$ has simple zeros at $\theta_0=0$ and π, implying transverse intersections of the two-dimensional stable and unstable manifolds; therefore, *two* Silnikov-type homoclinic orbits connect each hyperbolic equilibrium to itself (see fig. 5). As mentioned before, the orbits on both the stable and unstable manifolds passing through each perturbed equilibrium on the curve of equilibria are spirals. Therefore, the dynamics along the intersections of the stable and unstable manifolds of these equilibria falls into a class for which Devaney [10] has constructed Smale horseshoes, implying homoclinic chaos. A typical chaotic trajectory (see fig. 6) will spend most of its time near the two homoclinic orbits $\Gamma^{\pm}$ of the unperturbed problem, given in the original Maxwell–Bloch variables by

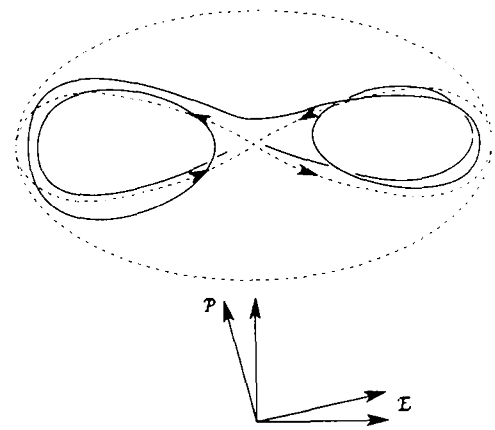

Fig. 6. Sketch of a typical chaotic trajectory showing intermittent switching.

$$\mathcal{E}=\pm 2\sqrt{K}\,\mathrm{sech}(\sqrt{K}\,t)\,,$$
$$\mathcal{P}=\mp 2K\,\mathrm{sech}(\sqrt{K}\,t)\tanh(\sqrt{K}\,t)\,,$$
$$\mathcal{D}=K[1-2\,\mathrm{sech}^2(\sqrt{K}\,t)]\,.$$

An initial condition starting near the unperturbed hyperbolic set $\mathcal{E}=0=\mathcal{P}$, $\mathcal{D}=K$, will follow a phase trajectory that stays in the vicinity of the homoclinic orbits Γ^+ and Γ^- (shown in fig. 1a) but switches intermittently between the right (+) and left (−) branches each time it passes near the point $\mathscr{E}=0=\mathscr{P}$, $\mathscr{S}=K$ (i.e., the origin in fig. 1a). Being associated to a Smale horseshoe construction, the intermittent switching shows extreme sensitivity to initial conditions. So, while this switching is deterministic in principle, it would be uncomputable and essentially random in practice.

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